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# Homoclinic solutions for a class of second-order Hamiltonian systems ☆

X.H. Tang [a,*], Xiaoyan Lin [a,b]

[a] School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410083, PR China
[b] Department of Mathematics, Huaihua College, Huaihua, Hunan 418008, PR China

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ABSTRACT

A new result for existence of homoclinic orbits is obtained for the second-order Hamiltonian systems $\ddot{u}(t) - L(t)u(t) + \nabla[W_1(t,u(t)) - W_2(t,u(t))] = f(t)$, where $t \in \mathbb{R}$, $u \in \mathbb{R}^n$ and $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $f \in C(\mathbb{R}, \mathbb{R}^n)$ are not necessary periodic in $t$. This result generalizes and improves some existing results in the literature.


## 1. Introduction

Consider the second-order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ and $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$. As usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u(t) \not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

The existence and multiplicity of homoclinic orbits for second-order Hamiltonian systems have been extensively investigated in many recent papers via critical point theory, see, e.g., [1–8,10–21]. For (1.1), if $L(t)$ and $W(t,x)$ are $T$-periodic in $t$, Rabinowitz [14] showed the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of (1.1). Analogous results for general Hamiltonian systems were obtained by Felmer et al. [7], Izydorek and Janczewska [10], Tang and Xiao [18] and Coti Zelati, Ekeland and Sere [19]. The related results can refer to [13] in the case where $L(t)$ and $W(t,x)$ are either independent of $t$.

If $L(t)$ and $W(t,x)$ are neither periodic in $t$, the problem of existence of homoclinic orbits for (1.1) is quite different from the ones just described, because of lack of compactness of the Sobolev embedding. In [15], Rabinowitz and Tanaka studied (1.1) without a periodicity assumption, both for $L$ and $W$ and obtained the following theorem by using a variant of the Mountain Pass theorem without the Palais–Smale condition.

**Theorem A.** *(See [15].) Assume that $L$ and $W$ satisfy the following conditions*:

(L) *$L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, (0,\infty))$ such that $l(t) \to +\infty$ as $|t| \to \infty$ and*

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* Corresponding author.
*E-mail address:* tangxh@mail.csu.edu.cn (X.H. Tang).

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$$\big(L(t)x, x\big) \geqslant l(t)|x|^2$$

*for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$;

(W1) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ *and there is a constant* $\mu > 2$ *such that*

$$0 < \mu W(t,x) \leqslant \big(x, \nabla W(t,x)\big)$$

*for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n \setminus \{0\}$;

(W2) $|\nabla W(t,x)| = o(|x|)$ *as* $|x| \to 0$ *uniformly with respect to* $t \in \mathbb{R}$;

(W3) *There is a* $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ *such that*

$$\big|W(t,x)\big| + \big|\nabla W(t,x)\big| \leqslant \big|\overline{W}(x)\big|$$

*for all* $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$.

*Then system* (1.1) *possesses a nontrivial homoclinic solution.*

By introducing a new compact imbedding theorem, Omana and Willem [11] showed that the Palais–Smale condition is satisfied and used the usual Mountain Pass theorem to prove the above Theorem A. In addition, Ding [5] and Ou and Tang [12] generalized Theorem A under more assumptions on $L$.

Motivated by the above paper [3,5,11,12,15], we will improve and generalize Theorem A along another direction. To this end, we consider the following general second-order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = f(t), \tag{HS}$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L$ is the same as (1.1), $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^n$ satisfy:

(H1) $W(t,x) = W_1(t,x) - W_2(t,x)$, $W_1, W_2 \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, and there is an $R > 0$ such that

$$\frac{1}{l(t)}\big|\nabla W(t,x)\big| = o\big(|x|\big), \quad \text{as } x \to 0,$$

uniformly in $t \in (-\infty, -R] \cup [R, +\infty)$;

(H2) There are two constants $\mu > 2$ and $\nu \in [0, 2^{-1}\mu - 1)$ such that

$$\frac{\mu\nu}{\mu - 2}\big(L(t)x, x\big) < \mu W_1(t,x) \leqslant \big(x, \nabla W_1(t,x)\big) + \nu\big(L(t)x, x\big)$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n \setminus \{0\}$;

(H3) $W_2(t,0) \equiv 0$ and there is a constant $\varrho \in [2, \mu)$ such that

$$W_2(t,x) \geqslant 0, \qquad \big(x, \nabla W_2(t,x)\big) \leqslant \varrho W_2(t,x)$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$;

(H4) $f \in C(\mathbb{R}, \mathbb{R}^n)$ and

$$\int_{\mathbb{R}} \big(L^{-1}(t)f(t), f(t)\big)\,dt < \begin{cases} \frac{(\mu-2-2\nu)^2}{2(\mu-1)^2}\big[\frac{\mu-2-2\nu}{2(\mu-2)}\big]^{2/(\mu-2)}\sqrt{l_*}, & \text{if } M > \frac{\mu-2-2\nu}{2(\mu-1)(\mu-2)}, \\ 2\big[\frac{\mu-2-2\nu}{2(\mu-2)} - M\big]^2\sqrt{l_*}, & \text{if } M \leqslant \frac{\mu-2-2\nu}{2(\mu-1)(\mu-2)}, \end{cases}$$

where $L^{-1}(t)$ is the inverse matrix of $L(t)$ and

$$l_* = \inf_{t\in\mathbb{R}} l(t), \qquad M = \sup\left\{\frac{W_1(t,x)}{l(t)} \;\middle|\; t \in \mathbb{R},\ x \in \mathbb{R}^n,\ |x| = 1\right\}. \tag{1.2}$$

In this paper, we will prove the following theorem.

**Theorem 1.1.** *Assume that* $L$, $W$ *and* $f$ *satisfy* (L) *and* (H1)–(H4)*. Then system* (HS) *possesses a nontrivial homoclinic solution.*

**Remark 1.1.** Obviously, for (1.1), i.e. (HS) with $W_2(t,x) \equiv 0$ and $f(t) \equiv 0$, Theorem 1.1 still improves Theorem A by relaxing conditions (W1) and (W2) (see (H1) and (H2)) and removing condition (W3).

## 2. Preliminaries

Let

$$E = \left\{ u \in W^{1,2}(\mathbb{R}, \mathbb{R}^n) \;\Big|\; \int_{\mathbb{R}} \left[ |\dot{u}(t)|^2 + \left(L(t)u(t), u(t)\right) \right] dt < +\infty \right\}$$

and for $u \in E$, let

$$\|u\| = \left\{ \int_{\mathbb{R}} \left[ |\dot{u}(t)|^2 + \left(L(t)u(t), u(t)\right) \right] dt \right\}^{1/2}.$$

Then $E$ is a Hilbert space on the above norm.

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}} \left[ -W_1(t, u(t)) + W_2(t, u(t)) + \left(f(t), u(t)\right) \right] dt. \tag{2.1}$$

Then $I \in C^1(E, \mathbb{R})$ and one can easily check that

$$I'(u)v = \int_{\mathbb{R}} \left[ \left(\dot{u}(t), \dot{v}(t)\right) + \left(L(t)u(t), v(t)\right) + \left(\nabla W_2(t, u(t)), v(t)\right) - \left(\nabla W_1(t, u(t)), v(t)\right) + \left(f(t), v(t)\right) \right] dt. \tag{2.2}$$

Furthermore, the critical points of $I$ in $E$ are classical solution of (HS) with $u(\pm\infty) = 0$.

We will obtain a critical point of $I$ by using a standard version of the Mountain Pass theorem. Since the minimax characterization it provides the critical value is important for what follows. Therefore, we state this theorem precisely.

**Lemma 2.1.** *(See [14].) Let $E$ be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the Palais–Smale condition. If $I$ satisfies the following conditions*:

(i) $I(0) = 0$;
(ii) *There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geqslant \alpha$;*
(iii) *There exists $e \in E \setminus \bar{B}_\rho(0)$ such that $I(e) \leqslant 0$, then $I$ possesses a critical value $c \geqslant \alpha$ given by*

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I\big(g(s)\big),$$

*where $B_\rho(0)$ is an open ball in $E$ of radius $\rho$ about at $0$, and*

$$\Gamma = \left\{ g \in C\big([0, 1], E\big):\ g(0) = 0,\ g(1) = e \right\}.$$

**Lemma 2.2.** *Let $a > 0$ and $u \in H^1(\mathbb{R}, \mathbb{R}^n)$. Then for every $t \in \mathbb{R}$, the following inequality holds*:

$$|u(t)| \leqslant (2a)^{-1/2} \left( \int_{t-a}^{t+a} |u(s)|^2 \, ds \right)^{1/2} + \sqrt{\frac{a}{6}} \left( \int_{t-a}^{t+a} |\dot{u}(s)|^2 \, ds \right)^{1/2}. \tag{2.3}$$

**Proof.** Fix $t \in \mathbb{R}$. For every $\tau \in \mathbb{R}$,

$$|u(t)| \leqslant |u(\tau)| + \left| \int_\tau^t \dot{u}(s) \, ds \right|. \tag{2.4}$$

Set

$$\varphi(s) = \begin{cases} s + a - t, & t - a \leqslant s \leqslant t, \\ t + a - s, & t \leqslant s \leqslant t + a. \end{cases}$$

Integrating (2.4) over $[t - a, t + a]$ and using the Hölder inequality, we obtain

$$
\begin{aligned}
2a\left|u(t)\right| &\leqslant \int_{t-a}^{t+a}\left|u(\tau)\right|d\tau + \int_{t-a}^{t+a}\left|\int_{\tau}^{t}\dot{u}(s)\,ds\right|d\tau \\
&\leqslant \int_{t-a}^{t+a}\left|u(\tau)\right|d\tau + \int_{t-a}^{t}\int_{\tau}^{t}\left|\dot{u}(s)\right|ds\,d\tau + \int_{t}^{t+a}\int_{t}^{\tau}\left|\dot{u}(s)\right|ds\,d\tau \\
&= \int_{t-a}^{t+a}\left|u(s)\right|ds + \int_{t-a}^{t}(s+a-t)\left|\dot{u}(s)\right|ds + \int_{t}^{t+a}(t+a-s)\left|\dot{u}(s)\right|ds \\
&= \int_{t-a}^{t+a}\left|u(s)\right|ds + \int_{t-a}^{t+a}\varphi(s)\left|\dot{u}(s)\right|ds \\
&\leqslant (2a)^{1/2}\left(\int_{t-a}^{t+a}\left|u(s)\right|^2ds\right)^{1/2} + \left(\int_{t-a}^{t+a}\left[\varphi(s)\right]^2ds\right)^{1/2}\left(\int_{t-a}^{t+a}\left|\dot{u}(s)\right|^2ds\right)^{1/2} \\
&= (2a)^{1/2}\left(\int_{t-a}^{t+a}\left|u(s)\right|^2ds\right)^{1/2} + \frac{a}{3}\sqrt{6a}\left(\int_{t-a}^{t+a}\left|\dot{u}(s)\right|^2ds\right)^{1/2},
\end{aligned}
$$

which implies that (2.3) holds. The proof is complete. □

**Corollary 2.1.** *Let $u \in H^1(\mathbb{R}, \mathbb{R}^n)$. Then for every $t \in \mathbb{R}$, the following inequality holds*:

$$
\left|u(t)\right| \leqslant \left[\int_{t-1}^{t+1}\left(\left|\dot{u}(s)\right|^2 + \left|u(s)\right|^2\right)ds\right]^{1/2}. \tag{2.5}
$$

**Proof.** Let $a = 1$ in (2.3). Then we obtain

$$
\left|u(t)\right| \leqslant \frac{\sqrt{2}}{2}\left[\left(\int_{t-1}^{t+1}\left|u(s)\right|^2ds\right)^{1/2} + \left(\int_{t-1}^{t+1}\left|\dot{u}(s)\right|^2ds\right)^{1/2}\right].
$$

It follows from the above and the inequality $(\sqrt{a}+\sqrt{b})/2 \leqslant \sqrt{(a+b)/2}$ that (2.5) holds. The proof is complete. □

**Lemma 2.3.** *For $u \in H^1(\mathbb{R}, \mathbb{R}^n)$,*

$$
\|u\|_{L^\infty(\mathbb{R})} \leqslant \frac{\sqrt{2}}{2}\|u\|_{H^1(\mathbb{R},\mathbb{R}^n)} = \frac{\sqrt{2}}{2}\left[\int_{\mathbb{R}}\left(\left|\dot{u}(s)\right|^2 + \left|u(s)\right|^2\right)ds\right]^{1/2}; \tag{2.6}
$$

*and for $u \in E$,*

$$
\|u\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{\sqrt{2\sqrt{l_*}}}\|u\| = \frac{1}{\sqrt{2\sqrt{l_*}}}\left\{\int_{\mathbb{R}}\left[\left|\dot{u}(s)\right|^2 + \left(L(s)u(s), u(s)\right)\right]ds\right\}^{1/2}, \tag{2.7}
$$

$$
\left|u(t)\right| \leqslant \left\{\int_{t}^{\infty}\frac{1}{\sqrt{l(s)}}\left[\left|\dot{u}(s)\right|^2 + \left(L(s)u(s), u(s)\right)\right]ds\right\}^{1/2}, \quad t \in \mathbb{R}, \tag{2.8}
$$

*and*

$$
\left|u(t)\right| \leqslant \left\{\int_{-\infty}^{t}\frac{1}{\sqrt{l(s)}}\left[\left|\dot{u}(s)\right|^2 + \left(L(s)u(s), u(s)\right)\right]ds\right\}^{1/2}, \quad t \in \mathbb{R}. \tag{2.9}
$$

**Proof.** Since $u \in H^1(\mathbb{R}, \mathbb{R}^n)$, it follows that

$$
\int_{\mathbb{R}}\left(\left|\dot{u}(t)\right|^2 + \left|u(t)\right|^2\right)dt < \infty,
$$

and so

$$\lim_{r\to\infty}\int\limits_{|t|\geqslant r}\left(\left|\dot u(t)\right|^2+\left|u(t)\right|^2\right)dt=0.$$

Combining this with (2.5), we have $\lim_{|t|\to\infty}|u(t)|=0$. Hence, if $u\in H^1(\mathbb{R},\mathbb{R}^n)$, then there exists a $t^*\in(-\infty,\infty)$ such that

$$\left|u\left(t^*\right)\right|=\max_{t\in\mathbb{R}}\left|u(t)\right|. \tag{2.10}$$

Choose two sequences $\{t_k\}$ and $\{t_{-k}\}$ such that

$$\cdots<t_{-3}<t_{-2}<t_{-1}<t^*<t_1<t_2<t_3<\cdots,$$
$$\lim_{k\to\infty}t_k=+\infty,\qquad \lim_{k\to\infty}t_{-k}=-\infty,$$

and

$$\lim_{k\to\infty}\left|u(t_k)\right|=\lim_{k\to\infty}\left|u(t_{-k})\right|=0.$$

Note that

$$\left|u\left(t^*\right)\right|^2=\left|u(t_k)\right|^2-2\int\limits_{t^*}^{t_k}\left(u(s),\dot u(s)\right)ds \tag{2.11}$$

and

$$\left|u\left(t^*\right)\right|^2=\left|u(t_{-k})\right|^2+2\int\limits_{t_{-k}}^{t^*}\left(u(s),\dot u(s)\right)ds. \tag{2.12}$$

From (2.11) and (2.12), we have

$$\begin{aligned}\left|u\left(t^*\right)\right|^2&=\frac{1}{2}\left(\left|u(t_k)\right|^2+\left|u(t_{-k})\right|^2\right)-\int\limits_{t^*}^{t_k}\left(u(s),\dot u(s)\right)ds+\int\limits_{t_{-k}}^{t^*}\left(u(s),\dot u(s)\right)ds\\&\leqslant\frac{1}{2}\left(\left|u(t_k)\right|^2+\left|u(t_{-k})\right|^2\right)+\int\limits_{t_{-k}}^{t_k}\left|u(s)\right|\left|\dot u(s)\right|ds\\&\leqslant\frac{1}{2}\left(\left|u(t_k)\right|^2+\left|u(t_{-k})\right|^2\right)+\frac{1}{2}\int\limits_{t_{-k}}^{t_k}\left(\left|\dot u(s)\right|^2+\left|u(s)\right|^2\right)ds,\quad k\in\mathbb{N}.\end{aligned}$$

Let $k\to\infty$ in the above, we obtain

$$\left|u\left(t^*\right)\right|^2\leqslant\frac{1}{2}\int\limits_{-\infty}^{\infty}\left(\left|\dot u(s)\right|^2+\left|u(s)\right|^2\right)ds,$$

which implies that (2.6) holds.

For $u\in E$, we have by (2.11) and (2.12),

$$\begin{aligned}\left|u\left(t^*\right)\right|^2&\leqslant\frac{1}{2}\left(\left|u(t_k)\right|^2+\left|u(t_{-k})\right|^2\right)+\int\limits_{t_{-k}}^{t_k}\left|u(s)\right|\left|\dot u(s)\right|ds\\&\leqslant\frac{1}{2}\left(\left|u(t_k)\right|^2+\left|u(t_{-k})\right|^2\right)+\frac{1}{2}\int\limits_{t_{-k}}^{t_k}\frac{1}{\sqrt{l(s)}}\left(\left|\dot u(s)\right|^2+l(s)\left|u(s)\right|^2\right)ds\\&\leqslant\frac{1}{2}\left(\left|u(t_k)\right|^2+\left|u(t_{-k})\right|^2\right)+\frac{1}{2}\int\limits_{t_{-k}}^{t_k}\frac{1}{\sqrt{l(s)}}\left[\left|\dot u(s)\right|^2+\left(L(s)u(s),u(s)\right)\right]ds\end{aligned}$$

$$\leqslant \frac{1}{2}\left(\left|u(t_k)\right|^2+\left|u(t_{-k})\right|^2\right)+\frac{1}{2\sqrt{l_*}}\int_{t_{-k}}^{t_k}\left[\left|\dot{u}(s)\right|^2+\left(L(s)u(s),u(s)\right)\right]ds,\quad k\in\mathbb{N}.$$

Let $k\to\infty$ in the above, we obtain

$$\left|u\left(t^*\right)\right|^2\leqslant\frac{1}{2\sqrt{l_*}}\int_{-\infty}^{\infty}\left[\left|\dot{u}(s)\right|^2+\left(L(t)u(t),u(t)\right)\right]ds,$$

which implies that (2.7) holds.

For any $t\in\mathbb{R}$, choose $k\in\mathbb{N}$ such that $t_{-k}<t<t_k$. Then we have

$$\left|u(t)\right|^2=\left|u(t_k)\right|^2-2\int_t^{t_k}\left(u(s),\dot{u}(s)\right)ds \tag{2.13}$$

and

$$\left|u(t)\right|^2=\left|u(t_{-k})\right|^2+2\int_{t_{-k}}^{t}\left(u(s),\dot{u}(s)\right)ds. \tag{2.14}$$

By (2.13), we have

$$\begin{aligned}\left|u(t)\right|^2&\leqslant\left|u(t_k)\right|^2+2\int_t^{t_k}\left|u(s)\right|\left|\dot{u}(s)\right|ds\\&\leqslant\left|u(t_k)\right|^2+\int_t^{t_k}\frac{1}{\sqrt{l(s)}}\left(\left|\dot{u}(s)\right|^2+l(s)\left|u(s)\right|^2\right)ds\\&\leqslant\left|u(t_k)\right|^2+\int_t^{t_k}\frac{1}{\sqrt{l(s)}}\left[\left|\dot{u}(s)\right|^2+\left(L(s)u(s),u(s)\right)\right]ds,\quad\text{for large } k\in\mathbb{N}.\end{aligned}$$

Let $k\to\infty$ in the above, we obtain

$$\left|u(t)\right|^2\leqslant\int_t^{\infty}\frac{1}{\sqrt{l(s)}}\left[\left|\dot{u}(s)\right|^2+\left(L(s)u(s),u(s)\right)\right]ds,$$

which implies that (2.8) holds.

Similarly, (2.9) can been proved by using (2.14) instead of (2.13). The proof is complete. □

**Lemma 2.4.** *Assume that* (H2) *and* (H3) *hold. Then for every $t\in\mathbb{R}$, the following inequalities hold*:

$$W_1(t,x)\leqslant\left[W_1\left(t,\frac{x}{|x|}\right)-\frac{\nu}{\mu-2}\left(L(t)\frac{x}{|x|},\frac{x}{|x|}\right)\right]|x|^\mu+\frac{\nu}{\mu-2}\left(L(t)x,x\right),\quad \textit{if } 0<|x|\leqslant 1, \tag{2.15}$$

$$W_1(t,x)\geqslant\left[W_1\left(t,\frac{x}{|x|}\right)-\frac{\nu}{\mu-2}\left(L(t)\frac{x}{|x|},\frac{x}{|x|}\right)\right]|x|^\mu+\frac{\nu}{\mu-2}\left(L(t)x,x\right),\quad \textit{if } |x|\geqslant 1, \tag{2.16}$$

*and*

$$W_2(t,x)\leqslant W_2\left(t,\frac{x}{|x|}\right)|x|^\varrho,\quad \textit{if } |x|\geqslant 1. \tag{2.17}$$

**Proof.** Set $\phi(s)=s^{-\mu}W_1(t,sx)$. Then by (H2), we have

$$\begin{aligned}\phi'(s)&=-\mu s^{-\mu-1}W_1(t,sx)+s^{-\mu}\left(\nabla W_1(t,sx),x\right)\\&=s^{-\mu-1}\left[-\mu W_1(t,sx)+\left(\nabla W_1(t,sx),sx\right)\right]\\&\geqslant-\nu s^{1-\mu}\left(L(t)x,x\right),\quad s>0.\end{aligned} \tag{2.18}$$

If $s \geqslant 1$, then it follows that

$$\phi(1) \leqslant \phi(s) + \frac{\nu}{\mu - 2}\big(1 - s^{2-\mu}\big)\big(L(t)x, x\big),$$

which implies that (2.15) holds. If $0 < s \leqslant 1$, then it follows from (2.18) that

$$\phi(1) \geqslant \phi(s) + \frac{\nu}{\mu - 2}\big(1 - s^{2-\mu}\big)\big(L(t)x, x\big),$$

which implies that (2.16) holds. By a similar fashion, we can prove that (2.17) holds. The proof is complete. □

## 3. Proof of the theorem

**Proof of Theorem 1.1.** It is clear that $I(0) = 0$. We firstly show that $I$ satisfies the Palais–Smale condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $c > 0$ such that

$$\big|I(u_k)\big| \leqslant \frac{c(\mu - 2)}{2\mu}, \qquad \big\|I'(u_k)\big\|_{E^*} \leqslant (\mu - 1)c, \quad \text{for } k \in \mathbb{N}. \tag{3.1}$$

From (2.1), (2.2), (3.1), (H2) and (H3), we obtain

$$\begin{aligned}
\frac{c(\mu-2)}{\mu} + \frac{2(\mu-1)c}{\mu}\|u_k\| &\geqslant 2I(u_k) - \frac{2}{\mu}I'(u_k)u_k \\
&= \frac{\mu-2}{\mu}\|u_k\|^2 + 2\int_{\mathbb{R}}\Big[W_2\big(t, u_k(t)\big) - \frac{1}{\mu}\big(\nabla W_2\big(t, u_k(t)\big), u_k(t)\big)\Big]\,dt \\
&\quad - 2\int_{\mathbb{R}}\Big[W_1\big(t, u_k(t)\big) - \frac{1}{\mu}\big(\nabla W_1\big(t, u_k(t)\big), u_k(t)\big)\Big]\,dt + \frac{2(\mu-1)}{\mu}\int_{\mathbb{R}}\big(f(t), u_k(t)\big)\,dt \\
&\geqslant \frac{\mu-2}{\mu}\|u_k\|^2 - \frac{2(\mu-1)}{\mu}\bigg(\int_{\mathbb{R}}\big(L^{-1}(t)f(t), f(t)\big)\,dt\bigg)^{1/2}\bigg(\int_{\mathbb{R}}\big(L(t)u_k(t), u_k(t)\big)\,dt\bigg)^{1/2} \\
&\geqslant \frac{\mu-2}{\mu}\|u_k\|^2 - \frac{2(\mu-1)}{\mu}\bigg(\int_{\mathbb{R}}\big(L^{-1}(t)f(t), f(t)\big)\,dt\bigg)^{1/2}\|u_k\| \\
&= \frac{\mu-2}{\mu}\|u_k\|^2 - \frac{2(\mu-1)\Lambda}{\mu}\|u_k\|, \quad k \in \mathbb{N},
\end{aligned}$$

where

$$\Lambda = \bigg(\int_{\mathbb{R}}\big(L^{-1}(t)f(t), f(t)\big)\,dt\bigg)^{1/2}.$$

It follows that there exists an $A > 0$ such that

$$\|u_k\| \leqslant A, \quad \text{for } k \in \mathbb{N}. \tag{3.2}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in $E$. For the any given number $\varepsilon > 0$, by (H1), we can choose $\eta > 0$ such that

$$\big|\nabla W(t, x)\big| \leqslant \varepsilon l(t)|x|, \quad \text{for } |t| \geqslant R \text{ and } |x| < \eta. \tag{3.3}$$

Since $l(t) \to \infty$, we can also choose a $T > R$ such that

$$l(t) \geqslant \frac{A^4}{\eta^4}, \quad |t| \geqslant T. \tag{3.4}$$

By (2.8), (3.2) and (3.4), we have

$$\big|u_k(t)\big|^2 \leqslant \int_t^\infty \frac{1}{\sqrt{l(s)}}\big[\big|\dot{u}_k(s)\big|^2 + \big(L(s)u_k(s), u_k(s)\big)\big]\,ds$$

$$
\begin{aligned}
&\leqslant \frac{\eta^2}{A^2}\int_t^{\infty}\big[\big|\dot u_k(s)\big|^2+\big(L(s)u_k(s),u_k(s)\big)\big]\,ds\\
&\leqslant \frac{\eta^2}{A^2}\|u_k\|^2\\
&\leqslant \eta^2, \quad \text{for } t\geqslant T,\ k\in\mathbb{N}.
\end{aligned}
\tag{3.5}
$$

Similarly, we have

$$\big|u_k(t)\big|^2\leqslant \eta^2, \quad \text{for } t\leqslant -T,\ k\in\mathbb{N}. \tag{3.6}$$

Noting that $u_k \rightharpoonup u_0$ in $E$, it is easy to verify that $u_k(t)$ converse to $u_0(t)$ pointwise for all $t\in\mathbb{R}$. Hence, we have by (3.5) and (3.6)

$$\big|u_0(t)\big|\leqslant \eta, \quad \text{for } t\in(-\infty,-T]\cup[T,+\infty). \tag{3.7}$$

Since $l(t)\geqslant c>0$ on $[-T,T]=J$, the operator defined by $S: E\to H^1(J): u\to u|_J$ is a linear continuous map. So $u_k\rightharpoonup u_0$ in $H^1(J)$. Sobolev's theorem (see e.g. [9]) implies that $u_k\to u_0$ uniformly on $J$, so there is a $k_0\in\mathbb{N}$ such that

$$\int_{-T}^{T}\big|\nabla W\big(t,u_k(t)\big)-\nabla W\big(t,u_0(t)\big)\big|\big|u_k(t)-u_0(t)\big|\,dt<\varepsilon, \quad \text{for } k\geqslant k_0. \tag{3.8}$$

On the other hand, it follows from (3.3), (3.5), (3.6) and (3.7) that

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus[-T,T]}\big|\nabla W\big(t,u_k(t)\big)-\nabla W\big(t,u_0(t)\big)\big|\big|u_k(t)-u_0(t)\big|\,dt\\
&\quad\leqslant \int_{\mathbb{R}\setminus[-T,T]}\big(\big|\nabla W\big(t,u_k(t)\big)\big|+\big|\nabla W\big(t,u_0(t)\big)\big|\big)\big(\big|u_k(t)\big|+\big|u_0(t)\big|\big)\,dt\\
&\quad\leqslant \varepsilon\int_{\mathbb{R}\setminus[-T,T]} l(t)\big(\big|u_k(t)\big|+\big|u_0(t)\big|\big)\big(\big|u_k(t)\big|+\big|u_0(t)\big|\big)\,dt\\
&\quad\leqslant 2\varepsilon\int_{\mathbb{R}\setminus[-T,T]} l(t)\big(\big|u_k(t)\big|^2+\big|u_0(t)\big|^2\big)\,dt\\
&\quad\leqslant 2\varepsilon\int_{\mathbb{R}\setminus[-T,T]}\big[\big(L(t)u_k(t),u_k(t)\big)+\big(L(t)u_0(t),u_0(t)\big)\big]\,dt\\
&\quad\leqslant 2\varepsilon\big(\|u_k\|^2+\|u_0\|^2\big)\\
&\quad\leqslant 4\varepsilon A^2, \quad k\in\mathbb{N}.
\end{aligned}
\tag{3.9}
$$

Combining (3.8) with (3.9) we get

$$\int_{\mathbb{R}}\big|\nabla W\big(t,u_k(t)\big)-\nabla W\big(t,u_0(t)\big)\big|\big|u_k(t)-u_0(t)\big|\,dt\to 0, \quad \text{as } k\to\infty. \tag{3.10}$$

It follows from (2.2) that

$$
\begin{aligned}
\big(I'(u_k)-I'(u_0)\big)(u_k-u_0)=\|u_k-u_0\|^2&-\int_{\mathbb{R}}\big(\nabla W\big(t,u_k(t)\big)-\nabla W\big(t,u_0(t)\big),u_k(t)-u_0(t)\big)\,dt\\
&+\int_{\mathbb{R}}\big(f(t),u_k(t)-u_0(t)\big)\,dt.
\end{aligned}
\tag{3.11}
$$

This shows that $u_k\to u_0$ in $E$. Hence, $I$ satisfies the Palais–Smale condition.

We now show that there exist constants $\rho$ and $\alpha>0$ such that $I$ satisfies the assumption (ii) of Lemma 2.1 with these constants. Choose $\delta\in(0,1]$ such that

$$\frac{\mu-2-2\nu}{2(\mu-2)}\delta-M\delta^{\mu-1}=\max_{x\in[0,1]}\bigg(\frac{\mu-2-2\nu}{2(\mu-2)}x-Mx^{\mu-1}\bigg).$$

Then

$$\frac{\mu-2-2\nu}{2(\mu-2)}\delta - M\delta^{\mu-1} = \begin{cases} \frac{\mu-2-2\nu}{2(\mu-1)}\left[\frac{\mu-2-2\nu}{2(\mu-2)}\right]^{1/(\mu-2)}, & \text{if } M > \frac{\mu-2-2\nu}{2(\mu-1)(\mu-2)}, \\ \frac{\mu-2-2\nu}{2(\mu-2)} - M, & \text{if } M \leqslant \frac{\mu-2-2\nu}{2(\mu-1)(\mu-2)}. \end{cases} \tag{3.12}$$

If $\|u\| = \sqrt{2\sqrt{l_*}}\delta = \rho$, then it follows from (2.7) that $|u(t)| \leqslant \delta \leqslant 1$ for $t \in \mathbb{R}$. By (1.2) and (2.15), we have

$$\begin{aligned}
\int_{\mathbb{R}} W_1\big(t, u(t)\big)\,dt &= \int_{\{t\in\mathbb{R}\,|\,u(t)\neq 0\}} W_1\big(t, u(t)\big)\,dt \\
&\leqslant \int_{\{t\in\mathbb{R}\,|\,u(t)\neq 0\}} \left[W_1\left(t, \frac{u(t)}{|u(t)|}\right)\big|u(t)\big|^\mu + \frac{\nu}{\mu-2}\big(L(t)u(t), u(t)\big)\right] dt \\
&\leqslant \int_{\mathbb{R}} \left[Ml(t)\big|u(t)\big|^\mu + \frac{\nu}{\mu-2}\big(L(t)u(t), u(t)\big)\right] dt \\
&\leqslant \int_{\mathbb{R}} \left[M\delta^{\mu-2}l(t)\big|u(t)\big|^2 + \frac{\nu}{\mu-2}\big(L(t)u(t), u(t)\big)\right] dt \\
&\leqslant \left(M\delta^{\mu-2} + \frac{\nu}{\mu-2}\right)\int_{\mathbb{R}} \big(L(t)u(t), u(t)\big)\,dt.
\end{aligned} \tag{3.13}$$

Set

$$\alpha = \sqrt{2\sqrt{l_*}}\delta\left[\sqrt{2\sqrt{l_*}}\left(\frac{\mu-2-2\nu}{2(\mu-2)}\delta - M\delta^{\mu-1}\right) - \Lambda\right]. \tag{3.14}$$

It follows from (H4) and (3.12) that $\alpha > 0$. Hence, from (2.1), (H3), (3.13), (3.14) and Hölder inequality, we have

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 + \int_{\mathbb{R}} \big[W_2\big(t, u(t)\big) - W_1\big(t, u(t)\big) + \big(f(t), u(t)\big)\big]\,dt \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}} \big|\dot{u}(t)\big|^2\,dt + \left(\frac{1}{2} - \frac{\nu}{\mu-2} - M\delta^{\mu-2}\right)\int_{\mathbb{R}} \big(L(t)u(t), u(t)\big)\,dt + \int_{\mathbb{R}} \big(f(t), u(t)\big)\,dt \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}} \big|\dot{u}(t)\big|^2\,dt + \left(\frac{1}{2} - \frac{\nu}{\mu-2} - M\delta^{\mu-2}\right)\int_{\mathbb{R}} \big(L(t)u(t), u(t)\big)\,dt \\
&\quad - \left(\int_{\mathbb{R}} \big(L^{-1}(t)f(t), f(t)\big)\,dt\right)^{1/2}\left(\int_{\mathbb{R}} \big(L(t)u(t), u(t)\big)\,dt\right)^{1/2} \\
&\geqslant \left(\frac{1}{2} - \frac{\nu}{\mu-2} - M\delta^{\mu-2}\right)\int_{\mathbb{R}} \big[\big|\dot{u}(t)\big|^2 + \big(L(t)u(t), u(t)\big)\big]\,dt - \Lambda\left(\int_{\mathbb{R}} \big[\big|\dot{u}(t)\big|^2 + \big(L(t)u(t), u(t)\big)\big]\,dt\right)^{1/2} \\
&= \left(\frac{1}{2} - \frac{\nu}{\mu-2} - M\delta^{\mu-2}\right)\|u\|^2 - \Lambda\|u\| \\
&= \sqrt{2\sqrt{l_*}}\delta\left[\sqrt{2\sqrt{l_*}}\left(\frac{\mu-2-2\nu}{2(\mu-2)}\delta - M\delta^{\mu-1}\right) - \Lambda\right] \\
&= \alpha.
\end{aligned} \tag{3.15}$$

(3.15) shows that $\|u\| = \rho$ implies that $I(u) \geqslant \alpha$.

Finally, it remains to show that $I$ satisfies the assumption (iii) of Lemma 2.1. Set

$$a_1 = \max\big\{W_2(t,x)\,\big|\, t \in [-2, 2],\ x \in \mathbb{R}^n,\ |x| = 1\big\}$$

and

$$a_2 = \max\big\{W_2(t,x)\,\big|\, t \in [-2, 2],\ x \in \mathbb{R}^n,\ |x| \leqslant 1\big\}.$$

Then by (H3) and (2.17), $0 \leqslant a_1 \leqslant a_2 < \infty$ and

$$W_2(t,x) \leqslant a_1|x|^\varrho + a_2, \quad \text{for } (t,x) \in [-2,2] \times \mathbb{R}^n. \tag{3.16}$$

Take $\omega \in E$ such that

$$\left|\omega(t)\right| = \begin{cases} 1, & \text{for } |t| \leqslant 1, \\ 0, & \text{for } |t| \geqslant 2, \end{cases} \tag{3.17}$$

and $|\omega(t)| \leqslant 1$ for $|t| \in (1, 2]$. For $\zeta > 1$, by (2.16) and (3.17), we have

$$\int_{-1}^{1} W_1\big(t, \zeta\omega(t)\big)\,dt \geqslant |\zeta|^{\mu} \int_{-1}^{1} \left[ W_1\big(t, \omega(t)\big) - \frac{\nu}{\mu - 2}\big(L(t)\omega(t), \omega(t)\big) \right] dt \geqslant 2m|\zeta|^{\mu}, \tag{3.18}$$

where

$$m = \min_{-1 \leqslant t \leqslant 1,\, |x|=1} \left[ W_1(t, x) - \frac{\nu}{\mu - 2}\big(L(t)x, x\big) \right] > 0,$$

see (H2). By (2.1), (3.16), (3.17), (3.18), (H2) and (H3), we have for $\zeta > 1$,

$$\begin{aligned} I(\zeta\omega) &= \frac{1}{2}\|\zeta\omega\|^2 + \int_{\mathbb{R}} \left[ W_2\big(t, \zeta\omega(t)\big) - W_1\big(t, \zeta\omega(t)\big) + \zeta\big(f(t), \omega(t)\big) \right] dt \\ &\leqslant \frac{|\zeta|^2}{2}\|\omega\|^2 + \int_{-2}^{2} \left[ W_2\big(t, \zeta\omega(t)\big) + \zeta\big(f(t), \omega(t)\big) \right] dt - \int_{-1}^{1} W_1\big(t, \zeta\omega(t)\big)\,dt \\ &\leqslant \frac{|\zeta|^2}{2}\|\omega\|^2 + a_1|\zeta|^{\varrho} \int_{-2}^{2} \left|\omega(t)\right|^{\varrho} dt + 4a_2 + \zeta \int_{-2}^{2} \big(f(t), \omega(t)\big)\,dt - 2m|\zeta|^{\mu}. \end{aligned} \tag{3.19}$$

Since $\mu > \varrho$ and $m > 0$, (3.19) implies that there exists $\xi > 1$ such that $\|\xi\omega\| > \rho$ and $I(\xi\omega) < 0$. Set $e(t) = \xi\omega(t)$. Then $e \in E$, $\|e\| = \|\xi\omega\| > \rho$ and $I(e) = I(\xi\omega) < 0$. By Lemma 2.1, $I$ possesses a critical value $d \geqslant \alpha$ given by

$$d = \inf_{g \in \Gamma} \max_{s \in [0,1]} I\big(g(s)\big), \tag{3.20}$$

where

$$\Gamma = \big\{ g \in C\big([0, 1], E\big):\ g(0) = 0,\ g(1) = e \big\}.$$

Hence, there exists $u^* \in E$ such that

$$I\big(u^*\big) = d \quad \text{and} \quad I'\big(u^*\big) = 0. \tag{3.21}$$

Then function $u^*$ is a desired classical solution of (HS). Since $d > 0$, $u^*$ is a nontrivial homoclinic solution even if $f(t) = 0$. $\square$

**Remark 3.1.** In this paper, we cannot prove that $\dot{u}(t) \to 0$ as $t \to \pm\infty$ for the homoclinic solution $u(t)$ of (HS). In fact, to the best of our knowledge, all the papers mentioned in Section 1 claimed that $\dot{u}(t) \to 0$ as $t \to \pm\infty$ if $u \in E$ is the critical point of the functional $I(u)$, but they did not prove this claim. It is routine to verify the above claim if $L(t)$ and $W(t, x)$ are bounded in $t$. However, we find that it seems be difficult to prove the above claim if $L(t)$ or $W(t, x)$ is not bounded in $t$. So we think it is also open to prove the above claim in this case.

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